Poincaré cycle of a multibox Ehrenfest urn model with directed transport

Yee-Mou Kao
National Center for High-Performance Computing, No. 21, Nan-ke 3rd Road, Hsin-Shi, Tainan County 744, Taiwan, Republic of China

Pi-Gang Luan
Institute of Optical Sciences, National Central University, Chung-Li 320 Taiwan, Republic of China

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We propose a generalized Ehrenfest urn model of many urns arranged periodically along a circle. The evolution of the urn model system is governed by a directed stochastic operation. Method for solving an *N*-ball, *M*-urn problem of this model is presented. The evolution of the system is studied in detail. We find that the average number of balls in a certain urn oscillates several times before it reaches a stationary value. This behavior seems to be a peculiar feature of this directed urn model. We also calculate the Poincaré cycle, i.e., the average time interval required for the system to return to its initial configuration. The result indicates that the fundamental assumption of statistical mechanics holds in this system.

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I. INTRODUCTION

Physical laws governing the microscopic processes are mostly reversible in time. In macroscopic world, however, people often experience time-irreversible phenomena in their daily life. To understand why the reversible microscopic processes lead to irreversible macroscopic manifestations one refers to the *Poincaré theorem*, which states that a system having a finite energy and confined to a finite volume will, after a sufficient long time—the so-called *Poincaré cycle*, return to an arbitrarily small neighborhood of almost any given initial state [1]. The key point is to note that the typical value of a Poincaré cycle for even a moderate-sized system is far beyond the meaningful time scale one can measure or experience, thus the irreversiblity is realized.

Usually to describe a macroscopic system, one has to know only a few parameters, such as volume, pressure, and temperature. However, to describe the same system in terms of its microscopic constituents, one has to deal with a large number of parameters, such as the momenta and positions of a huge amount of particles, which are impossible to calculate in practice. Based on this reason together with the fact that the macroscopic laws are insensitive to the microscopic details (of system history), it is natural for people to adopt the probability (ensemble) description in statistical mechanics, which deals with the equilibrium state (a macroscopic state that has stationary value of state parameters) of a macroscopic system. In this kind of description the macroscopic quantities are defined as the ensemble average of their microscopic correspondences. This definition connects the microscopic and macroscopic worlds.

To study how a system approaches its equilibrium state one also uses probability description, where the evolution of the system is treated as a stochastic process. One famous model for simulating such a process was proposed by Ehrenfest one century ago [2], which is an *N*-ball, 2-urn problem. In the beginning, *N* numbered balls are distributed arbitrarily in either urn *A* or urn *B*. At each time step one ball is picked out at random and then put into the other urn. This simple model can be exactly solved to give an explicit Poincaré cycle. This model was then generalized by several authors to mimic more complicated situations encountered in real physical phenomena [3–5]. An attractive feature of these urn model problems is that they are easy to formulate, but not always easy to solve. The solutions obtained have, therefore, sometimes led to new mathematical techniques and insights [6–9]. Recently, some new urn models were proposed and solved analytically or numerically. Their results provide very good descriptions on granular and glass systems [10–14].

In this paper, we obtain the exact solution of a generalized urn model. Hereafter, we call it as ''periodic urn model.'' In this model, one considers *N* distinguishable balls that are distributed in *M* urns. These *M* urns are arranged along a circle and numbered one by one to form a cycle, that is, we define the $(M+1)$th urn as the first urn (See Fig. 1). To begin with, the initial distribution of the *N* balls in the *M* urns is given by $|m_{1,0},m_{2,0},\ldots,m_{M,0}\rangle \equiv |\mathbf{m}_0\rangle$, where $m_{i,0}$ is the number of balls in the *i*th urn at the start. At each time step one ball is picked out of the *N* balls such that every ball has an equal probability of being picked up. The ball is then

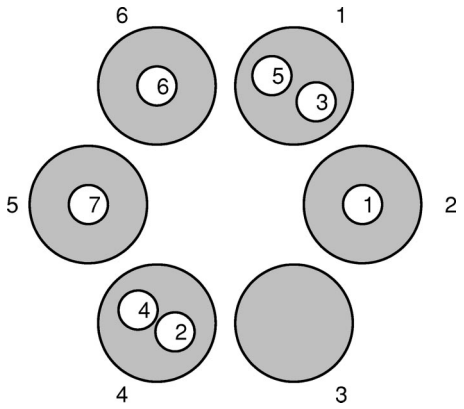

FIG. 1. Arrangement of the numbered urns and balls in our periodic urn model. The gray disks represent the urns and the white disks represent the balls. Here we illustrate a configuration for a system with six urns and seven balls. The state vector for this configuration is $|\mathbf{m}\rangle = |2,1,0,2,1,1\rangle$.

placed into the next numbered urn. The state that the ith urn contains m_i balls is represented by $|m_1,m_2,\ldots,m_M\rangle \equiv|\mathbf{m}\rangle$, which we name it as *state vector*. Hereafter we call a distribution string $\mathbf{m}$ (without knowing the numbering of the balls) a *configuration* of the system. Otherwise, if we also know the location of each numbered ball, we call such a distribution a *microstate* of the system.

After s steps, the transition probability from state $|\mathbf{m}_0\rangle$ to state $|\mathbf{m}\rangle$ can be written as $\langle\mathbf{m}|P^s|\mathbf{m}_0\rangle$, where P represents the operation in one step. The set of state vectors is taken to be orthonormal.

According to the above description, the transition probabilities corresponding to the sth step and the $(s-1)$th step satisfy the recursion relation

$$\langle m_1,m_2,\ldots,m_M|P^s|\mathbf{m}_0\rangle = \sum_{i=1}^{M}\frac{m_i+1}{N}\langle\ldots,m_i+1,m_{i+1}-1,\ldots|P^{s-1}|\mathbf{m}_0\rangle, \quad (1)$$

where $m_{M+1}=m_1$ as has been mentioned before. In addition, any state that does not satisfy the constraint

$$m_1+m_2+\cdots+m_M=N \quad (2)$$

is an unphysical state and has null contribution in the sum. Hereafter, we will use Eq. (1) as the basis to derive all the results we want to know.

This paper is organized as follows. In the following section we calculate the average number of balls in an urn at any time. In Sec. III we introduce a generating function that has N variables and solve the problem completely. In Sec. IV the solution of the model will be applied to the calculation of the Poincaré cycle. Finally in Sec. V we give the summary of this paper.

II. AVERAGE NUMBER OF BALLS IN AN URN

The first thing we want to know is how many balls on average will appear in the ith urn after first s steps. We study the problem in the following four subsections.

A. Exact solution

We define the average (the expectation value) of a quantity A [which depends on the state vector $|\mathbf{m}\rangle$ at each step, written as $A(\mathbf{m})$] after s steps as

$$\langle A\rangle_s=\sum_{\{\mathbf{m}\}}A(\mathbf{m})\langle\mathbf{m}|P^s|\mathbf{m}_0\rangle, \quad (3)$$

where $\{\mathbf{m}\}$ include all the configurations satisfying constraint (2).

Let $A(\mathbf{m})=m_i$, then from Eqs. (1) and (3) we have

$$\begin{aligned}\langle m_i\rangle_s&=\sum_{\{\mathbf{m}\}}m_i\langle\mathbf{m}|P^s|\mathbf{m}_0\rangle\\&=\sum_{\{\mathbf{m}\}}\sum_{j=1}^{M}\frac{m_i(m_j+1)}{N}\\&\quad\times\langle\ldots,m_j+1,m_{j+1}-1,\ldots|P^{s-1}|\mathbf{m}_0\rangle\\&=\sum_{j=1}^{M}\left(\frac{\langle m_im_j\rangle_{s-1}}{N}-\frac{\langle m_j\rangle_{s-1}}{N}\delta_{j,i}+\frac{\langle m_j\rangle_{s-1}}{N}\delta_{j+1,i}\right)\\&=\sum_{j=1}^{M}\frac{\langle m_im_j\rangle_{s-1}}{N}-\frac{\langle m_i\rangle_{s-1}}{N}+\frac{\langle m_{i-1}\rangle_{s-1}}{N}\\&=\left(1-\frac{1}{N}\right)\langle m_i\rangle_{s-1}+\frac{\langle m_{i-1}\rangle_{s-1}}{N}, \quad (4)\end{aligned}$$

here we have used constraint (2).

Now we are ready to solve $\langle m_i\rangle_s$. Recurrence relation (4) can be written as

$$\mathcal{M}_s=P_{ave}\mathcal{M}_{s-1}, \quad (5)$$

where $\mathcal{M}_s$ is a $M\times1$ column vector defined by

$$\mathcal{M}_s=\begin{bmatrix}\langle m_1\rangle_s\\\langle m_2\rangle_s\\\vdots\\\langle m_M\rangle_s\end{bmatrix}, \quad (6)$$

and P_{ave} is a $M\times M$ matrix written as

$$P_{ave}=\begin{bmatrix}1-\frac{1}{N}&0&\cdots&\frac{1}{N}\\\frac{1}{N}&1-\frac{1}{N}&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&1-\frac{1}{N}\end{bmatrix}. \quad (7)$$

By means of recurrence relation (5), $\mathcal{M}_s$ can be deduced,

$$\mathcal{M}_s=P_{ave}^s\mathcal{M}_0, \quad (8)$$

where

$$\mathcal{M}_0=\begin{bmatrix}m_{1,0}\\m_{2,0}\\\vdots\\m_{M,0}\end{bmatrix} \quad (9)$$

represents the initial state. P_{ave}^s can be calculated if one knows the eigenvalues λ_m and eigenvectors Q_m of P_{ave}. They are given by (see Fig. 2)

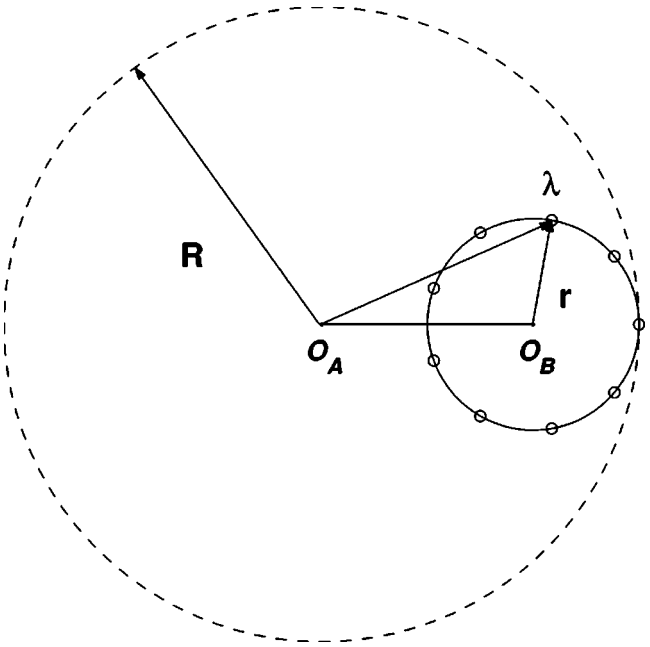

FIG. 2. Eigenvalues $\{\lambda_m\}$ of the matrix P_{ave} (represented by the tiny circles). Here $R=1$ and $r=1/N$ are the radii of two reference circles, and O_A and O_B are their centers, respectively. The eigenvalues of P_{ave} are distributed uniformly on the small reference circle centered at $O_B=(1-1/N,0)$.

$$\lambda_m=1-\frac{1}{N}+\frac{1}{N}q_m^*, \quad Q_m=\frac{1}{\sqrt{M}}\begin{bmatrix} q_m \\ q_m^2 \\ \vdots \\ q_m^M \end{bmatrix}, \tag{10}$$

where

$$q_m=\exp\left(\frac{2m\pi i}{M}\right), \quad m=1,2,\ldots,M. \tag{11}$$

Denote R as the $M\times M$ matrix of the eigenvectors Q_m,

$$R=[Q_1,Q_2,\ldots,Q_M] =\frac{1}{\sqrt{M}}\begin{bmatrix} q_1 & q_2 & \cdots & q_M \\ q_1^2 & q_2^2 & \cdots & q_M^2 \\ \vdots & \vdots & \ddots & \vdots \\ q_1^M & q_2^M & \cdots & q_M^M \end{bmatrix} \tag{12}$$

and Λ as the diagonal matrix of P_{ave}'s eigenvalues λ_m,

$$\Lambda=\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_M \end{bmatrix}, \tag{13}$$

then we obtain

$$P_{ave}^s=R\Lambda^s R^{-1}=R\Lambda^s R^\dagger=R\Lambda^s R^*, \tag{14}$$

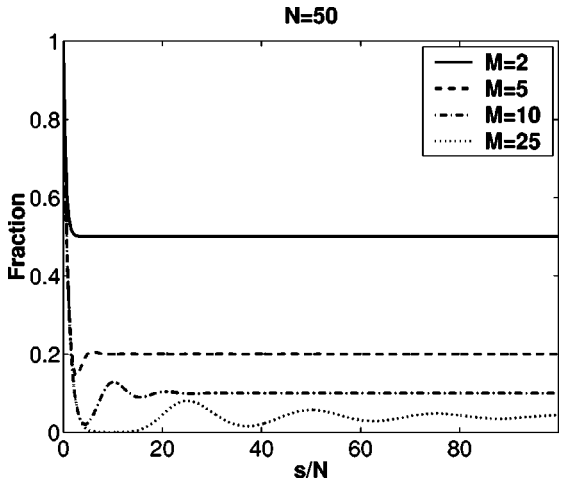

FIG. 3. Average number of balls in the first urn $\langle m_1\rangle_s$ as a function of time s, assuming initially all the balls are in the first urn. Here we plot ''fraction''$=\langle m_1\rangle_s/N$ for $N=50$ at $M=2,5,10,25$. As one can see, except for the $M=2$ case (which is the original Ehrenfest urn model), this mean value oscillates several times before it reaches a stationary value.

where we have used the following properties of R,

$$R=R^t, \quad (R^{-1})=R^\dagger=(R)^*, \quad R_{mn}=q_n^m=q_1^{mn}. \tag{15}$$

Now the average number of balls in the ith urn after s steps can be determined,

$$\langle m_i\rangle_s=\frac{1}{M}\sum_{j=1}^{M}\sum_{k=1}^{M}\sum_{l=1}^{M} q_j^i\lambda_j^s\delta_{jk}q_l^{-k}\langle m_l\rangle_0, =\frac{1}{M}\sum_{j=1}^{M}\sum_{l=1}^{M} q_1^{j(i-l)}\lambda_j^s m_{l,0}, \tag{16}$$

where $\langle m_l\rangle_0=m_{l,0}$ is the initial number of balls in the lth urn.

B. Numerical results

Let us now consider a simple example. Suppose initially all the N balls are in the first urn, that is,

$$m_{1,0}=N, \quad m_{2,0}=m_{3,0}=\cdots=m_{M,0}=0, \tag{17}$$

then according to Eq. (16), we have

$$\langle m_1\rangle_s=\frac{N}{M}\sum_{j=1}^{M}\lambda_j^s. \tag{18}$$

Figure 3 shows the results for $N=50$ at $M=2,5,10,25$. The $M=2$ case is the original Ehrenfest model, in which the average number of balls in the first urn decays to $N/2$ in a period of steps of order N. For any $M>2$ case, however, we observe that before the system arrives its true equilibrium (here we mean the value of $\langle m_i\rangle_s$ for each i does not change anymore), $\langle m_1\rangle_s$ undergoes several oscillations, which seems to be a unique feature of this model and have never been

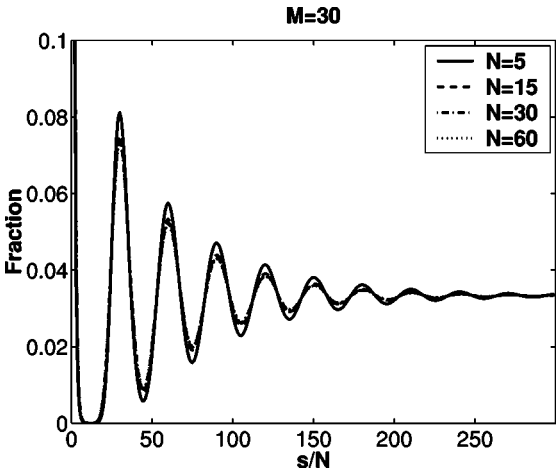

FIG. 4. Plot of fraction $=\langle m_1\rangle_s/N$ as a function of time s/N, assuming initially all the balls are in the first urn. Here $M=30$ and $N=5,15,30,60$. Except for the $N=5$ case (this N is too small), all the curves merge and become one. The visible range of fraction has been tuned to give a better illustration.

found in other kinds of urn models—to our knowledge. Furthermore, in Fig. 3 the $M=25$ case shows that before the appearance of the first peak of $\langle m_1\rangle_s$, there is a period during which $\langle m_1\rangle_s$ is almost zero. This phenomenon together with the oscillations mentioned before seem to be typical results when both M and N are large.

Some results for large M and large N are shown in Figs. 4 and 5. In Fig. 4 we plot the $\langle m_1\rangle_s/N$ curves for $M=30$ (which is large enough in practice) and $N=5,15,30,60$. As one can see, except for the $N=5$ case, the $\langle m_1\rangle_s/N$ curves corresponding to different N's merge and become one universal function of s/N. To understand this let us note that in the large N limit Eq. (18) can be approximated by

$$\langle m_1\rangle_s=\frac{N}{M}\sum_{j=1}^{M}\exp\left[\frac{s(q_j^{-1}-1)}{N}\right], \tag{19}$$

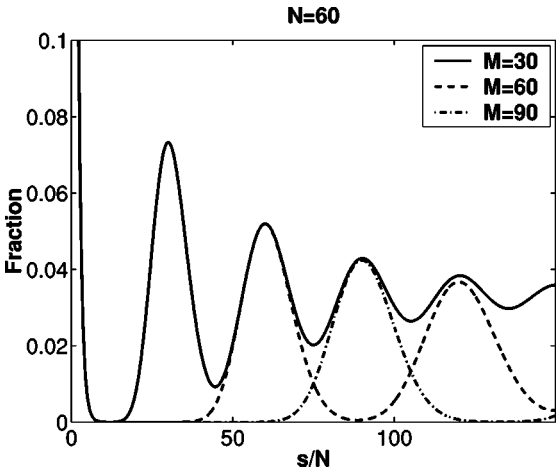

FIG. 5. Plot of fraction $=\langle m_1\rangle_s/N$ as a function of time s/N, assuming initially all the balls are in the first urn. Here $N=60$ and $M=30,60,90$. The first peaks of these three curves are located at $s/N=30,60,90$, respectively.

and hence $\langle m_1\rangle_s/N$ becomes a universal function of s/N for a fixed M. Figure 5 shows the results for $N=60$ and $M=30,60,90$. The local maxima of these $\langle m_1\rangle_s/N$ curves are located at $M, 2M, 3M, \ldots$, etc. We will explain this result in the following subsection by using an approximation of Eq. (19).

C. Two approximations

Starting from Eq. (19), we now derive two useful approximations of $\langle m_1\rangle_s/N$, which can help us understand the observations in Figs. 4 and 5. First, define

$$\tau\equiv\frac{s}{N}, \tag{20}$$

and expand the exponential functions in Eq. (19) as power series of τ, we have

$$\frac{\langle m_1\rangle_s}{N}=\frac{e^{-\tau}}{M}\sum_{j=1}^{M}\left(1+\tau q_j^{-1}+\frac{1}{2}\tau^2 q_j^{-2}+\cdots\right)$$
$$=e^{-\tau}\left[1+\frac{\tau^M}{M!}+\frac{\tau^{2M}}{(2M)!}+\cdots\right]. \tag{21}$$

In deriving Eq. (21), we have used the fact that

$$\sum_{j=1}^{M}q_j^{-k}=\sum_{j=1}^{M}q_1^{-jk}=\begin{cases}0 & \text{if} \quad k/M\neq\text{integer}\\ M & \text{if} \quad k/M=\text{integer.}\end{cases} \tag{22}$$

Note that the form of the jth term $(j>0)$ appearing in the last line of Eq. (21) is the same as the probability of n successful trials in a *Poisson process* [15]:

$$P(n)=\frac{e^{-\tau}\tau^n}{n!}. \tag{23}$$

Here $n=jM$, and $\tau=\langle n\rangle$ is the expectation value of n. This is not an accident and can be easily understood. The quantity $\langle m_1\rangle_s/N$ represents not only the average number of balls in the first urn divided by N, but also the probability of finding a certain ball, say, ball 1, in the first urn. For example, if $\langle m_1\rangle_s=N$, then the probability of finding ball 1 in the first urn is 1. At each time step, ball 1 has the probability of $p=1/N$ being picked out and moved to the next urn. Now since N is large, p is small. In this limit if we do the same operation s times (here we assume s is also large and define $\tau=ps=s/N$), then the probability for ball 1 to be moved n steps forward from the first urn is given by Eq. (23). Furthermore, since our system has a circulating property, the probability of finding ball 1 in the first urn after s steps consists of the following possibilities: (i) ball 1 has never been selected [the corresponding probability is $(1-p)^s\approx e^{-ps}=e^{-\tau}$], and (ii) it has been picked up M times, and (iii) it has been chosen $2M$ times, and so on. The summation of all these contributions gives us the expression of the last line of Eq. (21).

Applying the *saddle-point method* and the *Stirling formula* [15] to each $e^{-\tau}\tau^{jM}/(jM)!$ term, we have

$$\frac{e^{-\tau}\tau^{jM}}{(jM)!} \approx \frac{\left(\frac{jM}{e}\right)^{jM} e^{-(\tau-jM)^2/2jM}}{\sqrt{2\pi jM}\left(\frac{jM}{e}\right)^{jM}} = \frac{e^{-(\tau-jM)^2/2jM}}{\sqrt{2\pi jM}}, \tag{24}$$

and thus

$$\frac{\langle m_1\rangle_s}{N} \approx e^{-\tau} + \frac{e^{-(\tau-M)^2/2M}}{\sqrt{2\pi M}} + \frac{e^{-(\tau-2M)^2/4M}}{\sqrt{4\pi M}} + \cdots. \tag{25}$$

Equation (25) shows that (1) the $\langle m_1\rangle_s/N$ curve consists of an exponentially decaying term $e^{-\tau}$ and a series of Gaussian terms of different heights. They give a good qualitative description of the results observed in Figs. 4 and 5. (2) The center of mass of these N balls circulate through the M urns with a period $\Delta s = NM$, consistent with the definition of the model. (3) As time goes by, the distribution of these N balls becomes broader and broader, implying a diffusion effect. (4) The ''time difference'' between two successive maxima is $\Delta\tau = M$, whereas the standard deviation of the jth Gaussian term is $\sqrt{jM}$, thus if $j \ll M$, we have $\sqrt{jM} \ll \Delta\tau$, and the overlap between two successive Gaussian terms can be neglected.

If $\tau \ll M^2$, then in the above expression we have to include the terms only up to the order $j_{\max} \approx \tau/M \ll M$, and thus Eq. (25) gives not only a qualitative but also a good quantitative description of the $\langle m_1\rangle_s/N$ curve.

When τ becomes too large that the overlap between two successive Gaussian terms cannot be neglected, then Eq. (25) becomes useless. However, we can show that if τ becomes larger than $M^2/2\pi^2$, then another accurate approximation can be obtained. Note that Eq. (19) can be rewritten as

$$\frac{\langle m_1\rangle_s}{N} = \frac{1}{M}[1 + 2e^{-\tau(1-\cos\theta_1)}\cos(\tau\sin\theta_1) + 2e^{-\tau(1-\cos\theta_2)}\cos(\tau\sin\theta_2) + \cdots], \tag{26}$$

where we have defined

$$\theta_j = \frac{2\pi j}{M} = j\theta, \quad \theta = \frac{2\pi}{M}. \tag{27}$$

Remember that M is a large number, so

$$\sin\theta_1 \approx \theta, \quad 1-\cos\theta_1 \approx \frac{\theta^2}{2}. \tag{28}$$

Thus for $\tau \gg M^2/2\pi^2$ [i.e., $\tau(1-\cos\theta_1) \approx \tau\theta^2/2 \gg 1$] we have the approximation

$$\frac{\langle m_1\rangle_s}{N} \approx \frac{1 + 2e^{-\tau\theta^2/2}\cos(\tau\theta)}{M}. \tag{29}$$

D. Center of mass—the global consideration

Up to now we have been focusing our attention on only one single urn. Now we show that we can also understand the behaviors of the system in a global manner. First, define the ''phase angle'' of the kth urn (see Fig. 1) as

$$\phi_k = -(k-1)\theta. \tag{30}$$

Also, we define the ''center of mass'' (COM) of our N-ball, M-urn system as

$$x_{\rm COM} \equiv \frac{\sum_{k=1}^{M} e^{i\phi_k}\langle m_k\rangle_s}{\sum_{k=1}^{M}\langle m_k\rangle_s} = \sum_{k=1}^{M} e^{i\phi_k}\frac{\langle m_k\rangle_s}{N}. \tag{31}$$

According to Eq. (31) $x_{\rm COM}$ is in general a complex number, say, $z = re^{i\phi}$. Here $r = |z|$ satisfies $0 \leqslant r \leqslant 1$, and ϕ is the phase angle of z. Variation of ϕ with respect to $\tau = s/N$ represents how fast on average these N balls circulate through the M urns, and the norm r gives us the information of the distribution of the N balls. For example, if every urn has N/M balls, then $r = 0$. Substitute Eqs. (18) and (30) into Eq. (31) and use Eqs. (10) and (11), we get

$$\begin{aligned} x_{\rm COM} &= \frac{1}{M}\sum_{j=1}^{M}\sum_{k=1}^{M} q_1^{(j-1)(k-1)}\lambda_j^s \\ &= \lambda_1^s \\ &\approx e^{-\tau\theta^2/2}e^{-i\tau\theta}, \end{aligned} \tag{32}$$

hence

$$r = e^{-\tau\theta^2/2}, \quad \phi = -\tau\theta. \tag{33}$$

Here we see that the COM curve is approximately described by a spiral circulating inside a unit circle. The angular frequency of this circulating motion with respect to $\tau = s/N$ is $-\theta = -2\pi/M$ (clockwise), consistent with both the definition of our model and the oscillation behaviors of the $\langle m_1\rangle_s/N$ curve discussed before. Furthermore, when $\tau > M^2/2\pi^2$, we have $r < e^{-1}$, which indicates that now these N balls are distributed in a wide extent, also consistent with the result of the second approximation in the last subsection. Some examples are illustrated in Figs. 6 and 7.

III. STATE MATRIX AND GENERATING FUNCTION

Now we calculate $\langle \mathbf{m}|P^s|\mathbf{m}_0\rangle$—the transition probability from $|\mathbf{m}_0\rangle$ to $|\mathbf{m}\rangle$ after s steps. Once one knows the exact solution of $\langle \mathbf{m}|P^s|\mathbf{m}_0\rangle$, any quantity can be calculated explicitly.

Define

$$S_{\mathbf{m}\mathbf{m}'} = \langle \mathbf{m}|P|\mathbf{m}'\rangle, \tag{34}$$

then we have

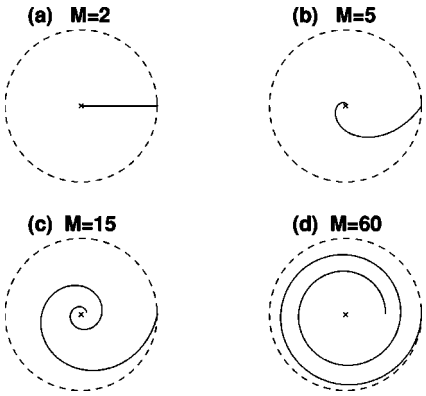

FIG. 6. Plot of the trace of the center of mass (COM) as a function of time step s, assuming initially all the balls are in the first urn. Here the curves are plotted for $s=0$ to $s=2MN$ with $M=2,5,15,60$. Each curve with large enough M ($M\geqslant 10$) has circulated the origin (the "x" symbol) of the complex plane twice after evolving $2MN$ steps.

$$\langle \mathbf{m}|P^s|\mathbf{m}_0\rangle=(S^s)_{\mathbf{m}\mathbf{m}_0}. \tag{35}$$

Here S is a $H_N^M\times H_N^M$ matrix, we name it as *state matrix*, and $|\mathbf{m}\rangle$ is a H_N^M column vector, here

$$H_N^M=C_{M-1}^{N+M-1}=\frac{(N+M-1)!}{N!(M-1)!}. \tag{36}$$

Like before, S^s can be calculated by means of its eigenvalues and eigenstates. According to Eq. (1), the matrix S has components

$$S_{\mathbf{m}\mathbf{m}'}=\sum_{i=1}^{M}\frac{m_i+1}{N}\times\delta_{m_1,m_1'}\delta_{m_2,m_2'}\cdots\delta_{m_i+1,m_i'}\delta_{m_{i+1}-1,m_{i+1}'}\cdots\delta_{m_M,m_M'}, \tag{37}$$

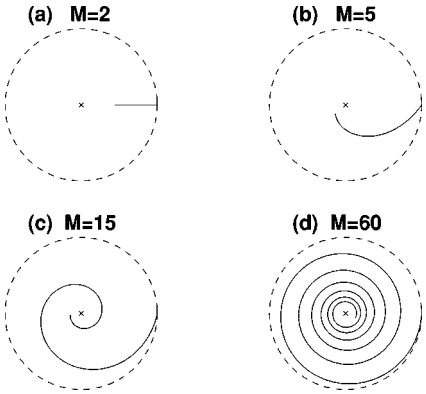

FIG. 7. These COM curves are plotted for $s=0$ to $s=M^2N/\pi^2$ with $M=2,5,15,60$. The norm of a COM in the complex plane becomes $e^{-2}=0.1353$ times smaller after evolving M^2N/π^2 steps if M is large enough ($M\geqslant 10$). Here the symbol "x" denotes the origin.

where $m_{M+1}=m_1$ have been assumed. The eigenvalue equation can be written as

$$\sum_{\{\mathbf{m}'\}} S_{\mathbf{m}\mathbf{m}'}\phi_{\mathbf{m}'}=\gamma\phi_{\mathbf{m}}, \tag{38}$$

or more explicitly

$$\sum_{i=1}^{M}\frac{m_i+1}{N}\phi_{m_1,m_2,\ldots,m_i+1,m_{i+1}-1,\ldots,m_M}=\gamma\phi_{m_1,m_2,\ldots,m_i,m_{i+1},\ldots,m_M}. \tag{39}$$

As in Eq. (1), $\phi_{\mathbf{m}}=0$ for any unphysical $\mathbf{m}$ [an $\mathbf{m}$ that does not satisfy Eq. (2)].

It is not an easy task to diagonalize S directly. Thus we adopt another strategy. We first construct a generating function for $\phi_{m_1,m_2,\ldots,m_M}$ and then transform the matrix eigenvalue equation (38) to its differential equation form. We find that the differential equation can be solved analytically.

By introducing variables $x_1,x_2,\ldots,x_M$, the generating function can be defined as

$$f(x_1,x_2,\ldots,x_M)\equiv\sum_{\{\mathbf{m}\}}\phi_{m_1,m_2,\ldots,m_M}x_1^{m_1}x_2^{m_2}\cdots x_M^{m_M}. \tag{40}$$

Hereafter, we also use the following expression:

$$f(\mathbf{X})=\sum_{\{\mathbf{m}\}}\phi_{\mathbf{m}}X^{\mathbf{m}}, \tag{41}$$

where $\mathbf{X}$ and $X^{\mathbf{m}}$ are defined by

$$\mathbf{X}\equiv\begin{bmatrix}x_1\\x_2\\\vdots\\x_M\end{bmatrix},\quad X^{\mathbf{m}}\equiv x_1^{m_1}x_2^{m_2}\cdots x_M^{m_M}. \tag{42}$$

To proceed further, note that $f(\mathbf{X})$ satisfies the following two relations:

$$\partial_{x_i}f(\mathbf{X})=\sum_{\{\mathbf{m}\}}(m_i+1)\phi_{m_1,m_2,\ldots,m_i+1,\ldots,m_M}X^{\mathbf{m}},$$

$$x_i f(\mathbf{X})=\sum_{\{\mathbf{m}\}}\phi_{m_1,m_2,\ldots,m_i-1,\ldots,m_M}X^{\mathbf{m}}, \tag{43}$$

as can be easily checked. Multiplying $X^{\mathbf{m}}$ on both sides of Eq. (39), summing over all $\{\mathbf{m}\}$, and using the results of Eq. (43), we get

$$\sum_{i=1}^{M}\frac{x_{i+1}}{N}\partial_{x_i}f(\mathbf{X})=\gamma f(\mathbf{X}), \tag{44}$$

or equivalently

$$\sum_{i=1}^{M} x_{i+1}\partial_{x_i}\ln[f(\mathbf{X})]=N\gamma, \tag{45}$$

which is the desired differential equation form of eigenvalue equation (39). Define

$$x_{q_j}=x_1q_j+x_2q_j^2+\cdots+x_Mq_j^M, \tag{46}$$

we find

$$\sum_{i=1}^{M} x_{i+1}\partial_{x_i}\ln(x_{q_j})=q_j^{-1}=q_j^*. \tag{47}$$

This implies that the complete solution of $\ln[f(\mathbf{X})]$ can be written as

$$\ln[f_{\mathbf{n}}(\mathbf{X})]=\sum_{j=1}^{M} n_j\ln(x_{q_j}), \tag{48}$$

which gives us

$$f_{\mathbf{n}}(\mathbf{X})=\prod_{j=1}^{M} x_{q_j}^{n_j}\equiv X_{\mathbf{q}}^{\mathbf{n}}. \tag{49}$$

Here $f_{\mathbf{n}}(\mathbf{X})$(a homogeneous Nth power function) and the eigenvalue $\gamma_{\mathbf{n}}$ are characterized by $\mathbf{n}=[n_1,n_2,\ldots,n_M]$ and $\mathbf{q}=[q_1,q_2,\ldots,q_M]$, satisfying

$$N=\sum_{j=1}^{M} n_j,\quad \gamma_{\mathbf{n}}=\frac{1}{N}\sum_{j=1}^{M} n_jq_j^*=\frac{\mathbf{n}\cdot\mathbf{q}^*}{N}. \tag{50}$$

Denoting the $\mathbf{n}$th eigenvector of S as $\phi(\mathbf{n})$, Eqs. (38) and (41) now become

$$\sum_{\{\mathbf{m}'\}} S_{\mathbf{m}\mathbf{m}'}\phi_{\mathbf{m}'}(\mathbf{n})=\gamma_{\mathbf{n}}\phi_{\mathbf{m}}(\mathbf{n}) \tag{51}$$

and

$$f_{\mathbf{n}}(\mathbf{X})=\sum_{\{\mathbf{m}\}} \phi_{\mathbf{m}}(\mathbf{n})X^{\mathbf{m}}=X_{\mathbf{q}}^{\mathbf{n}}. \tag{52}$$

To diagonalize S we first define an orthogonal transformation matrix U,

$$U_{\mathbf{mn}}=\phi_{\mathbf{m}}(\mathbf{n}), \tag{53}$$

where $\phi_{\mathbf{m}}(\mathbf{n})$ according to Eq. (52) is the coefficient of $X^{\mathbf{m}}$ that appears in the expansion of $f_{\mathbf{n}}(\mathbf{X})=X_{\mathbf{q}}^{\mathbf{n}}$.

We now are ready to solve the matrix U^{-1}. Multiplying $X^{\mathbf{m}}=x_1^{m_1}x_2^{m_2}\cdots x_M^{m_M}$ on both sides of

$$\sum_{\{\mathbf{n}\}} \phi_{\mathbf{m}}(\mathbf{n})U_{\mathbf{nl}}^{-1}=\sum_{\{\mathbf{n}\}} U_{\mathbf{mn}}U_{\mathbf{nl}}^{-1}=\delta_{\mathbf{ml}} \tag{54}$$

and summing over all possible $\{\mathbf{m}\}$, we get

$$\sum_{\{\mathbf{n}\}} f_{\mathbf{n}}(\mathbf{X})U_{\mathbf{nm}}^{-1}=\sum_{\{\mathbf{n}\}} X_{\mathbf{q}}^{\mathbf{n}}U_{\mathbf{nm}}^{-1}=X^{\mathbf{m}}. \tag{55}$$

Furthermore, define two vectors $\mathbf{Y}$ and $\mathbf{X_q}$ as

$$\mathbf{Y}=\begin{bmatrix} y_1\\ y_2\\ \vdots\\ y_M\end{bmatrix}\equiv\begin{bmatrix} q_1 & q_2 & \cdots & q_m\\ q_1^2 & q_2^2 & \cdots & q_m^2\\ \vdots & \vdots & \ddots & \vdots\\ q_1^M & q_2^M & \cdots & q_m^M\end{bmatrix}\begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_M\end{bmatrix}, \tag{56}$$

and

$$\mathbf{X_q}\equiv\begin{bmatrix} x_{q_1}\\ x_{q_2}\\ \vdots\\ x_{q_M}\end{bmatrix}, \tag{57}$$

we have

$$\mathbf{Y}=\sqrt{M}R\mathbf{X}. \tag{58}$$

Using the same notation and remember that $R^{-1}=R^*$, we find

$$\mathbf{X}=\frac{1}{\sqrt{M}}R^*\mathbf{Y}=\frac{1}{M}\mathbf{Y_{q^*}}. \tag{59}$$

These results further lead to

$$\begin{aligned} X^{\mathbf{m}}&=\frac{1}{M^N}(Y_{\mathbf{q}^*})^{\mathbf{m}}=\frac{1}{M^N}Y_{\mathbf{q}}^{\tilde{\mathbf{m}}}=\frac{1}{M^N}f_{\tilde{\mathbf{m}}}(\mathbf{Y}),\\ &=\frac{1}{M^N}\sum_{\{\mathbf{n}\}}\phi_{\mathbf{n}}(\tilde{\mathbf{m}})Y^{\mathbf{n}}=\frac{1}{M^N}\sum_{\{\mathbf{n}\}}\phi_{\mathbf{n}}(\tilde{\mathbf{m}})X_{\mathbf{q}}^{\mathbf{n}}\\ &=\frac{1}{M^N}\sum_{\{\mathbf{n}\}} f_{\mathbf{n}}(\mathbf{X})\phi_{\mathbf{n}}(\tilde{\mathbf{m}}). \end{aligned} \tag{60}$$

Comparing Eq. (60) with Eq. (55), we get

$$U_{\mathbf{nm}}^{-1}=\frac{1}{M^N}\phi_{\mathbf{n}}(\tilde{\mathbf{m}}), \tag{61}$$

where we have used the relations:

$$\begin{aligned} Y_{\mathbf{q}^*}^{\mathbf{m}}&=y_{q_1^*}^{m_1}y_{q_2^*}^{m_2}y_{q_3^*}^{m_3}\cdots y_{q_{M-1}^*}^{m_{M-1}}y_{q_M^*}^{m_M}\\ &=y_{q_{M-1}}^{m_1}y_{q_{M-2}}^{m_2}y_{q_{M-3}}^{m_3}\cdots y_{q_1}^{m_{M-1}}y_{q_M}^{m_M}\\ &=y_{q_1}^{m_{M-1}}y_{q_2}^{m_{M-2}}y_{q_3}^{m_{M-3}}\cdots y_{q_{M-1}}^{m_1}y_{q_M}^{m_M}\\ &=Y_{\mathbf{q}}^{\tilde{\mathbf{m}}}=f_{\tilde{\mathbf{m}}}(\mathbf{Y}), \end{aligned} \tag{62}$$

and defined $\tilde{\mathbf{m}}$ as

$$\tilde{\mathbf{m}}\equiv[m_{M-1},m_{M-2},m_{M-3},\ldots,m_1,m_M]. \tag{63}$$

Finally we obtain the desired solution of $\langle\mathbf{m}|P^s|\mathbf{m}_0\rangle$,

$$\langle \mathbf{m}|P^s|\mathbf{m}_0\rangle = (S^s)_{\mathbf{m}\mathbf{m}_0},$$
$$= (U\Gamma^s U^{-1})_{\mathbf{m}\mathbf{m}_0},$$
$$= \frac{1}{M^N}\sum_{\mathbf{m}'} \gamma_{\mathbf{m}'}^s \phi_{\mathbf{m}}(\mathbf{m}')\phi_{\mathbf{m}'}(\tilde{\mathbf{m}}_0), \quad (64)$$

where Γ is the eigenvalue matrix of S, which has components $\Gamma_{\mathbf{m}\mathbf{m}'} = \gamma_{\mathbf{m}}\delta_{\mathbf{m}\mathbf{m}'}$.

IV. THE POINCARÉ CYCLE

In this section, we study the Poincaré cycle of our periodic urn model. For simplicity we first consider the situation that initially all the N balls are stayed in the last urn, i.e.,

$$\mathbf{m}_0 = [0,0,0,\ldots,N].$$

From Eqs. (65) and (63) we have

$$\mathbf{m}_0 = \tilde{\mathbf{m}}_0. \quad (65)$$

Now we want to know how many time steps on average are required for all of the N balls to return to the last urn (the initial state). We thus have to calculate

$$\langle \mathbf{m}_0|P^s|\mathbf{m}_0\rangle = \frac{1}{M^N}\sum_{\mathbf{m}} \gamma_{\mathbf{m}}^s \phi_{\mathbf{m}_0}(\mathbf{m})\phi_{\mathbf{m}}(\tilde{\mathbf{m}}_0). \quad (66)$$

Recall that $\phi_{\mathbf{m}}(\mathbf{n})$ is nothing but the coefficient of $X^{\mathbf{m}} = x_1^{m_1}\cdots x_M^{m_M}$ appearing in the expansion of $f_{\mathbf{n}}(\mathbf{X}) = X_{\mathbf{q}}^{\mathbf{n}} = x_{q_1}^{n_1}\cdots x_{q_M}^{n_M}$. From Eqs. (52) and (65), we have

$$\phi_{\mathbf{m}_0}(\mathbf{m}) = 1, \quad (67)$$

$$\phi_{\mathbf{m}}(\tilde{\mathbf{m}}_0) = \frac{N!}{m_1!m_2!m_3!\cdots m_M!} \equiv \binom{N}{\mathbf{m}}, \quad (68)$$

and hence

$$\langle \mathbf{m}_0|P^s|\mathbf{m}_0\rangle = \frac{1}{M^N}\sum_{\mathbf{m}} \binom{N}{\mathbf{m}} \gamma_{\mathbf{m}}^s \equiv \mathcal{P}(s). \quad (69)$$

Here $\mathcal{P}(s)$ represents the transition probability for the system to return to the initial state after s steps. It does not preclude the possibility that the initial state has already been rearrived before.

Since the Poincaré cycle is defined as the time interval required for the event of first return to happen, so we have to do more calculations to extract what we really want. We define a function $\mathcal{Q}(s)$ as the probability for the event of first return to happen at the sth step. The Poincaré cycle can thus be defined as

$$\mathcal{P} = \sum_{s=0}^{\infty} s\mathcal{Q}(s). \quad (70)$$

By definition $\mathcal{Q}(s)$ relates to $\mathcal{P}(s)$ via the relation

$$\mathcal{P}(s) = \mathcal{Q}(s) + \sum_{k=1}^{s-1} \mathcal{Q}(k)\mathcal{P}(s-k), \quad (71)$$

and hence $\mathcal{Q}(s)$ can be calculated from $\mathcal{P}(s)$. To ease the calculation we now use again the generating function method. We first define two generating functions:

$$h(z) \equiv \sum_{s=1}^{\infty} \mathcal{P}(s)z^s, \quad g(z) \equiv \sum_{s=1}^{\infty} \mathcal{Q}(s)z^s, \quad (72)$$

and then we find from Eq. (71) that

$$g(z) = \frac{h(z)}{h(z)+1}. \quad (73)$$

These two generating functions also lead to

$$\sum_{s=0}^{\infty} s\mathcal{Q}(s) = \left(\frac{dg}{dz}\right)_{z=1}, \quad g' = \frac{h'}{(1+h)^2}, \quad (74)$$

which can determine the Poincaré cycle.

We now calculate $h(z)$. From Eqs. (69) and (82) we obtain

$$h(z) = \frac{1}{M^N}\sum_{\mathbf{m}} \binom{N}{\mathbf{m}} \sum_{s=1}^{\infty} (\gamma_{\mathbf{m}}z)^s,$$
$$= \frac{1}{M^N}\sum_{\mathbf{m}} \binom{N}{\mathbf{m}} \left(\frac{\gamma_{\mathbf{m}}z}{1-\gamma_{\mathbf{m}}z}\right). \quad (75)$$

Since we know from Eq. (50) that $\gamma_{\mathbf{m}_0} = 1$, thus when $z \to 1^-$, $h(z)$ becomes singular,

$$\lim_{z\to 1^-} h(z) = \frac{1}{M^N}\frac{z}{1-z} + \text{regular function}. \quad (76)$$

In this limit, we obtain

$$\lim_{z\to 1^-} g' = \lim_{z\to 1^-} \frac{h'}{(1+h)^2} = M^N, \quad (77)$$

which gives us the desired Poincaré cycle $\mathcal{P}$:

$$\mathcal{P} = \sum_{s=0}^{\infty} s\mathcal{Q}(s) = M^N. \quad (78)$$

To understand the meaning of this result, we refer to the *ergodic theorem* [1], which says that if one waits for a sufficiently long time, the locus of the representative point of a system will cover the entire accessible phase space. For our periodic urn model, the ''representative point'' corresponds to the microstate of the arrangement of balls, the ''accessible phase space'' is the set of total M^N microstates, and the ''locus'' means the evolution history of the system (see Fig. 1). The result that the Poincaré cycle equals the total number of microstates of the system is a strong indication that the fun-

damental assumption of statistical mechanics (equal probability of occurrence for each microstate) holds in this system.

What will be the Poincaré cycle if initially these N balls are not in a single urn? Let us denote the initial state by $\mathbf{d}$,

$$|\mathbf{d}\rangle = |d_1, d_2, \ldots, d_M\rangle. \tag{79}$$

Now we have

$$\mathcal{P}(s) = \langle \mathbf{d}|P^s|\mathbf{d}\rangle = \frac{1}{M^N} \sum_{\{\mathbf{m}\}} \gamma_{\mathbf{m}}^s \phi_{\mathbf{d}}(\mathbf{m}) \phi_{\mathbf{m}}(\tilde{\mathbf{d}}) \tag{80}$$

and

$$h(z) = \frac{1}{M^N} \sum_{\{\mathbf{m}\}} \phi_{\mathbf{d}}(\mathbf{m}) \phi_{\mathbf{m}}(\tilde{\mathbf{d}}) \left(\frac{\lambda_{\mathbf{m}} z}{1-\lambda_{\mathbf{m}} z} \right). \tag{81}$$

In general, it is difficult to calculate $\phi_{\mathbf{d}}(\mathbf{m})$ and $\phi_{\mathbf{m}}(\tilde{\mathbf{d}})$. However, we do not need to calculate them all. Remember that to determine the Poincaré cycle the knowledge of the asymptotic form of $h(z)$ near $z=1$ is enough. This is given by

$$\lim_{z \to 1^-} h(z) = \frac{\phi_{\mathbf{d}}(\mathbf{m}_0) \phi_{\mathbf{m}_0}(\tilde{\mathbf{d}})}{M^N} \frac{z}{1-z} + \text{regular function}. \tag{82}$$

Substituting

$$\phi_{\mathbf{d}}(\mathbf{m}_0) = \binom{N}{\mathbf{d}}, \quad \phi_{\mathbf{m}_0}(\tilde{\mathbf{d}}) = 1 \tag{83}$$

into (82), we find

$$\mathcal{P} = \lim_{z \to 1^-} \frac{h'}{(1+h)^2} = \frac{M^N}{\binom{N}{\mathbf{d}}}. \tag{84}$$

This result can be easily understood by considering the definitions of *configuration* and *microstate* (See Sec. I). There is only one microstate that corresponds to $|\mathbf{m}_0\rangle$, whereas there are

$$\binom{N}{\mathbf{d}} = \frac{N!}{d_1! d_2! \cdots d_M!} \tag{85}$$

microstates that correspond to $|\mathbf{d}\rangle$. Thus, on average the first return time for $|\mathbf{d}\rangle \to |\mathbf{d}\rangle$ becomes $\binom{N}{\mathbf{d}}^{-1}$ times of that of $|\mathbf{m}_0\rangle \to |\mathbf{m}_0\rangle$. Here we find conclusive evidence that the fundamental assumption of statistical mechanics on equal probability of occurrence for each microstate holds in this system.

V. SUMMARY

In this work we propose a generalized Ehrenfest urn model of many urns arranged periodically along a circle. We solve an N-ball, M-urn problem explicitly. The evolution of the system is studied, and the average number of balls in a certain urn at any time step is calculated. We find that this mean value oscillates several times before it arrives the stationary value. We also obtained the Poincaré cycle for two situations. The results indicate that the fundamental assumption of statistical mechanics holds in this system.

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